

Anomalous nonequilibrium Ising-Bloch bifurcation in discrete systems

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We investigate the self-organization of two-dimensional activator-inhibitor discrete bistable systems in the neighborhood of a nonequilibrium Ising-Bloch bifurcation. The system exhibits an anomalous transition—induced by discretization—whose signature is the coexistence of Ising and Bloch walls for selected values of the spatial coupling. After curvature reduction of Bloch walls, coexistence gives rise to a unique and striking spatiotemporal dynamics: Bloch walls drive the motion and Ising walls play the role of “extended defects” oriented along the background grid directions. Strong enough external noise asymptotically restores the scenario found in the continuum limit.

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INTRODUCTION

The nonequilibrium Ising-Bloch (NIB) transition has been identified as an important mechanism of pattern formation in reaction-diffusion [1–4], optical [6,7], and liquid-crystal systems [8,9]. It consists of a pitchfork bifurcation where a stationary (Ising) front loses stability to a pair of counterpropagating (Bloch) fronts. These Bloch fronts connect the same asymptotic states but they propagate in opposite directions, reflecting a nonvariational dynamics.

For NIB transitions in the continuum, the destabilization of an Ising front is initiated by a critical eigenvector that coincides with the translational (Goldstone) mode at the critical point [10]. The bifurcated fronts may produce complex spatiotemporal phenomena involving traveling domains and spontaneous nucleation of spiral waves followed by domain breakup [11–14]. This bifurcation is also closely related to chiral symmetry breaking in magnetic domain walls [15] and forced oscillatory systems [16].

Activator-inhibitor kinetics like the FitzHugh–Nagumo (FHN) model provide a canonical example: a diffusing autocatalytic species (activator) produces a second substance (inhibitor) in a different time scale, which in turn consumes the activator, thus introducing a negative feedback. For fast inhibitors and in the bistable regime, the dynamics exhibit Ising fronts. For sufficiently slow kinetics—in fact, for the activator-inhibitor time-scales ratio ϵ below a critical threshold ϵ_c —Bloch fronts are the stable interface. Structurally, Bloch fronts differ from an Ising front in that the inhibitor steplike profile is displaced with respect to that of the activator. The magnitude and the sign of the displacement determine the speed and direction of propagation, respectively. The velocity c of the bifurcated fronts follows (asymptotically) a square-root law: $c \propto \sqrt{\epsilon_c - \epsilon}$ [4].

In two spatial dimensions, in the Ising regime one finds Ising walls (IWs), and eventually the system converges to a uniform state due to curvature effects. In the Bloch regime, one observes Bloch walls (BWs) that give rise to spiral waves around pointlike defects [4,5].

This scenario changes drastically for discrete systems. In particular, an anomalous NIB transition was predicted recently for one-dimensional systems, whose signature is the *coexistence* of stable Ising and Bloch domain walls [17]. This fact has deep consequences for the organization of spatially coupled FHN cells, where discretization is crucial for a realistic description (e.g., of neuronal activity).

In this paper we investigate numerically the main consequences of an anomalous NIB transition in two-dimensional (2D) discrete activator-inhibitor systems. In particular, we consider the following dimensionless nongradient activator-inhibitor reaction-diffusion model of the FitzHugh–Nagumo type [1,18,19]:

$$\frac{du_{i,j}}{dt} = D(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}) + u_{i,j} - u_{i,j}^3 - v_{i,j},$$

$$\frac{dv_{i,j}}{dt} = \epsilon(u_{i,j} - av_{i,j}), \quad (1)$$

where $u_{i,j}, v_{i,j}$ are the field values at the (i, j) cell, ϵ is the ratio between the activator and inhibitor time scales, and D is the spatial coupling. Hereafter we choose $a=2$ such that Eqs. (1) describe a bistable medium with two linearly stable homogeneous solutions: $P_+ = (u_+, v_+)$ (up state) and $P_- = (u_-, v_-)$ (down state), and one unstable $P_0 = (0, 0)$ (saddle point). Here $u_+ = -u_- = \sqrt{1-a^{-1}}$ and $v_+ = -v_- = u_+/a$. Note that the up and down states are symmetric under parity $P_- = -P_+$, as a consequence of the system's invariance: $(u, v) \rightarrow (-u, -v)$.

In a previous work [17], we considered the one-dimensional version of (1). The main result is summarized in Fig. 1. The D versus ϵ parameter space is organized as follows: Ising fronts are stable below the D_H line (the locus of

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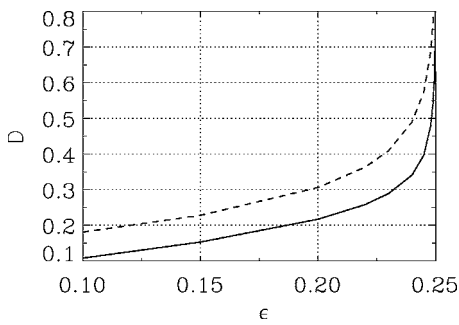


FIG. 1. Regions with different front regimes in the parameter space for the discrete FHN model in one dimension. Here $a=2$ and the solid line indicates $D_{SN}(\epsilon)$, the locus of the saddle-node bifurcation below which the IWs are the only stable interfaces. In the same way, the dashed line indicates $D_H(\epsilon)$, the locus of the Hopf bifurcation above which the BWs are the only stable interfaces. Between these two lines the coexistence region takes place.

a subcritical Hopf bifurcation). Stable Bloch fronts exist above D_{SN} where they emerge through twin saddle-node bifurcations. Both lines meet at $D=\infty$ only, at the point where the “continuous” NIB transition arises [4]: $\epsilon=\epsilon_c=1/a^2$. For finite D , i.e., when the system is discrete, there is a region of coexistence of Ising and Bloch fronts in the (infinite) wedge-like region between D_{SN} and D_H lines. The diagram indicates that a NIB transition can take place at constant ϵ by changing the spatial coupling D [20].

The bifurcation lines $D_{SN,H}$ emanate from the double-zero eigenvalue point located in the continuum limit, namely the “continuous” NIB bifurcation point. The degeneracy (an extra null eigenvalue) arises from the invariance under translation. The discretization, introduced through parameter D , breaks the translational symmetry, and together with ϵ fully unfold the double-zero point. As argued in [17], the FHN model (Fig. 1) exhibits one of the universal scenarios that arise from the inclusion of inhomogeneous terms into the normal form of the NIB bifurcation [21] (in a more abstract language referred to as parity-breaking or drift-pitchfork bifurcation).

ISING AND BLOCH DOMAIN-WALL REGIMES

For small spatial coupling—in fact for $D < D_{SN}$ —Ising fronts are the stable interfaces. Numerical solutions with random initial conditions show the spontaneous formation and coarsening of spatial homogeneous domains P_{\pm} separating by IWs. As flat fronts are stationary, the transient dynamics is controlled by the curvature of the fronts. In particular, curvature reduction leads to a regime of domain growth.

For $D > D_H$, numerical simulations in one and two spatial dimensions confirm that only Bloch domain walls are stable, and BWs spiral around pointlike defects in two spatial dimensions [6–8,22].

In both cases the dynamics resembles the traditional one before/after a continuous NIB transition. Similar structures and dynamics have been reported, for example, in the description of the ordering process of domains in optical parametric oscillators [7].

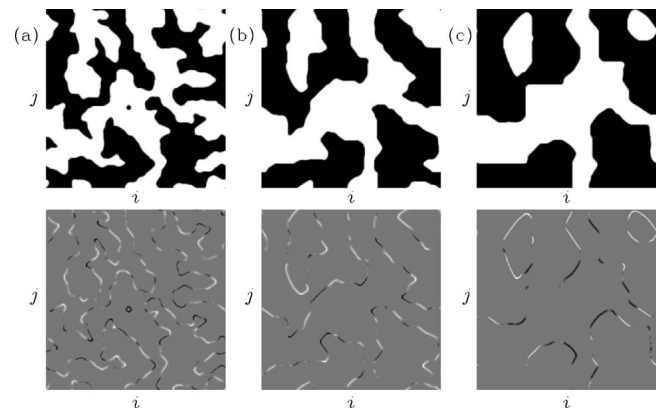


FIG. 2. Snapshot of $v_{i,j}$ (top) and $\eta_{i,j}$ (bottom) at different times observed in the coexistence regime: (a) $t=75$; (b) $t=150$; (c) $t=225$. Top: white (black) denotes the P_+ (P_-) state. Bottom: white, black, and gray correspond to positive, negative, and null values of $\eta_{i,j}$, respectively. The initial condition is random, $dt=0.00025$, and the values of the parameters are $D=0.23$ ($D_{SN}=0.217$), $\epsilon=0.2$, and $a=2$. Note that at short times curvature effects drive the dynamics and basically BWs are observed. After curvature reduction, IW and BW coexist macroscopically along the same front. Note that the IW’s orientation is restricted to the grid directions. Here the grid is 250×250 with periodic BC.

ISING-BLOCH COEXISTENCE

For $D_{SN}(\epsilon) < D < D_H(\epsilon)$, Ising and Bloch stable domain walls coexist. The system only nucleates phases up and down, but an additional feature of BWs in 2D is that the domain walls can emerge with opposite chirality in different spatial regions along the same wall. Even more, zero chirality is here also allowed. For a better characterization of the structures, we introduce the “chirality field:” $\eta_{i,j}=u_{i,j}-av_{i,j}$ which essentially measures deviation from stationarity, i.e., it differs from zero only at BW cores, having the sign of the relative front displacement.

In Fig. 2 we show snapshots of the v -pattern configuration at three times, for a typical realization of Eqs. (1) obtained through numerical simulation with random initial conditions. The chirality field η is also showed. Both fields must be considered simultaneously to discriminate BWs from IWs. The nature and dynamics of the observed structures is completely unique and striking: IWs coexist with BWs along the same front, with the IWs playing the role of “extended defects” separating BWs. Note that IWs here are dynamical structures which may grow or collapse, originating a dynamics. In particular, after curvature reduction of BWs, the observed structures are IWs separating BWs. During time evolution IWs are generated or shrunk, depending on the chirality of the BWs, which fix the sense of motion of the chiral arms. To illustrate this point, in Fig. 3 we show how an IW evolves during the “spiraling” as an extended core. In particular, we can appreciate a kind of “elastic scattering” between BWs of opposite chiralities, leading to a $\pi/2$ rotation of the IW’s direction. In Fig. 3 we have used nonflux (Neumann) boundary conditions (BC) in order to limit the growth of the IW. These BC preserve the translation and rotation motions of the Bloch fronts at the boundaries (they

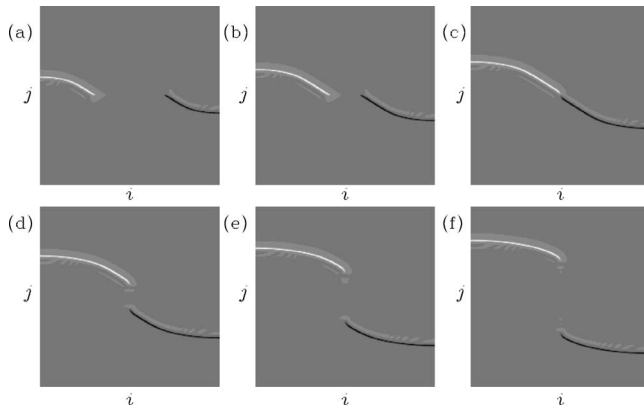


FIG. 3. Time evolution of a “spiral” in the coexistence region of a discrete system closed by nonflux boundary conditions. We show only the chirality field. The core of the spiral is an IW which changes size and orientation during time evolution in a periodic way, even though it preserves the “center-of-mass” core position. In (a) and (b) the IW is along the x axis, in (c) the core is basically a point, and in (d)–(f) the IW generated after collision is along the y direction. The values of the parameters are $\epsilon=0.2$, $D=0.26$ ($D_{SN}=0.217$), $dt=0.0005$; the grid is 210×210 and there is a 60 units time interval between the snapshots.

are not absorbed) leading to a periodic oscillation (both in length and orientation) of the IW while the arms of the spiral remain rotating.

As shown above, in the coexistence regime the dynamics is driven by the BWs, which control the domain growth. In particular, curvature reduction of IWs does not play any role. Another aspect of the dynamics is that BWs with different chirality can annihilate between them, producing domain evaporation. This mechanism is illustrated in Fig. 4, where we show snapshots of the pattern arising through numerical integration from a selected initial condition. Here we can observe again the coexistence of BWs and IWs for short time, as well as the interaction between both BWs, which eventually produces the evaporation of the up domain. Note

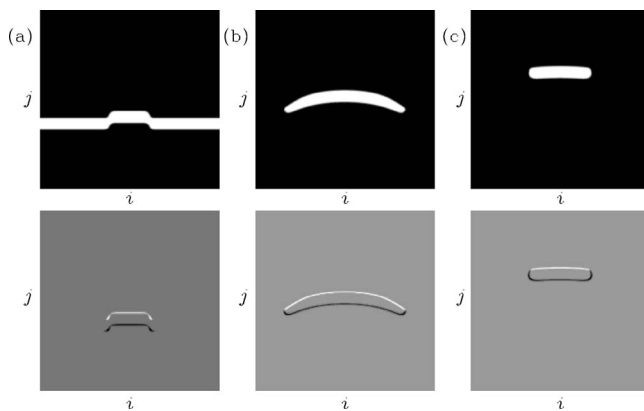


FIG. 4. Snapshot of $v_{i,j}$ (top) $\eta_{i,j}$ (bottom) in the coexistence regime. The initial conditions are two parallel IWs with a central “chiral” perturbation in each of them. For large time the up state eventually evaporates. (a) $t=3$; (b) $t=42$; (c) $t=75$. The values of the parameters are the same as in Fig. 3 with a grid 210×210 and $dt=0.00025$.

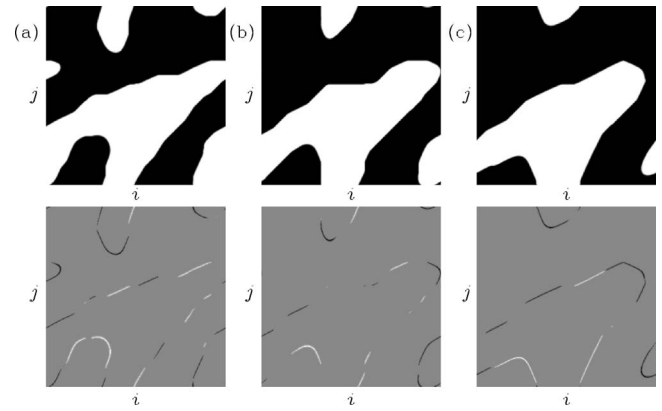


FIG. 5. Snapshot of $v_{i,j}$ (top) and $\eta_{i,j}$ (bottom) at different times observed in the coexistence regime for a nine-point spatial coupling with random initial conditions. The values of the parameters are $D=0.24$ ($D_{SN}=0.217$), $\epsilon=0.2$, $a=2$, and the times are: (a) $t=240$; (b) $t=320$, and (c) $t=400$. Note that the IW’s orientation is again restricted to the grid directions, including now the same diagonals of the spatial coupling. Here the grid is 250×250 with periodic BC and $dt=0.00025$.

that moving pointlike defects are formed during time evolution. In this case, and in contrast to Fig. 3, the parallel motion of BWs preserves the pointlike nature of the defects. The propagation of “traveling fragments” [like those in Figs. 4(b) and 4(c)] has been studied in Ref. [23] and modeled in terms of diffusion anisotropy.

We remark that in this region of parameter space the system is very sensitive to the background grid orientation: IWs are generated, collapsed and sustained only along the grid axes. This fact has deep consequences in the dynamics, because spiral cores lose their stability, and cannot hold the ends of the arms of BWs. The sensitivity of the IW with respect to the background directions (which is not observed outside the coexistence region) can be traced back to the basis of attraction of IWs in the one-dimensional case. In particular, IWs are very sensitive to core fluctuations in such a way that if $\{u_i, v_i\}$ is a stable Ising front in the coexistence region, the v -displaced profile $\{u_i, v_{i\pm 1}\}$ is an unstable structure that decays to a stable Bloch front [26]. Although this is a global perturbation of the core, stable under more localized perturbations, it is the kind of (chirality) fluctuations that fronts experiment in two spatial dimensions due to curvature effects [24]. These results are robust in the sense that other discretizations of the Laplacian, e.g., with a nine-point formula [25], originate a similar dynamics. Figure 5 shows some snapshots of pattern evolution for the nine-point coupling. Note that IWs can be formed also along diagonals, following the directions of the grid.

It is to be emphasized that coexistence does not imply equal stability, being the BW (IW) the stable (metastable) attractor of the system, as we expect from the weak stability of IWs. In particular, numerical simulations of Eqs. (1) with an additive source of white Gaussian noise indicate that the anomalous transition is observed for small noise intensities, while strong enough external noise (asymptotically) restores the continuouslike transition. In this case, coexistence is only observed during a transient, i.e., before the noise-assisted decay of the metastable fronts.

CONCLUSION

We have investigated the effects of discretization in a NIB transition for a 2D array of spatially coupled FHN cells. In particular, we have verified that, as reported in [17], discrete FHN systems have an intermediate regime where both Ising and Bloch fronts coexist.

In the coexistence regime, the system exhibits striking features in *two* spatial dimensions. IWs and BWs coexist, the former playing the role of extended defect cores. The robustness of these results has been tested with different discretizations of the Laplacian. In all the cases, IWs are only sustained along the grid axes, reflecting the poor stability of IWs against relative translation between both activator and inhibitor components.

The same scenario is observed when external fluctuations are included, although for strong enough external noise the coexistence takes place only during a transient, reflecting the metastable character of IWs in this region. Note that a complete route from IWs to BWs was presented at ϵ constant,

only by changing the strength of the spatial coupling. In this sense, discretization allows an “orthogonal” route (D varies, ϵ fixed) to observe a NIB transition, even in the presence of a strong external noise.

Finally, we want to remark that the scenario presented here for the FHN model should be found in other bistable systems undergoing NIB bifurcations. As claimed in our previous work [17], Ising-Bloch coexistence is a generic effect of discretization. Therefore, the two-dimensional dynamics studied in this paper should be observed in a wide range of discrete systems for which the FHN model serves as a canonical example.

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